

SUFFICIENT CONDITIONS FOR SOME CLASSES OF p-VALENTLY MEROMORPHIC FUNCTIONS

Dr. Deepaly Nigam

Abstract : We introduce two new classes $M_p(m,q; \alpha)$ and $N_p(m,q; \alpha)$ of meromorphic p -valent function in the punctured unit disk $U = \{Z : Z \in \mathbb{C} \text{ and } |Z| < 1\}$. Some theorems involving sufficient conditions for functions belong to these classes are obtained.

1. INTRODUCTION

Let M_p denote the class of functions $f(z)$ of the form

$$(1.1) \quad f(z) = \frac{1}{z^p} + \sum_{k=p}^{\infty} a_k z^k; \quad (p \in \mathbb{N} = \{1, 2, \dots\}),$$

which are analytic in punctured unit disk $D = U/\{0\}$ where $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

A function belonging to the class M_p is said to be meromorphically starlike of order α if and only if

$$(1.2) \quad \operatorname{Re} \left(\frac{-zf'(z)}{f(z)} \right) > \alpha; \quad (z \in D, 0 \leq \alpha < p).$$

We denote by $S_p^*(\alpha)$ the subclass of M_p consisting of functions which are meromorphically convex of order α if and only if

$$(1.3) \quad \operatorname{Re} \left[- \left(1 + \frac{zf''(z)}{f'(z)} \right) \right] > \alpha; \quad (z \in D, 0 \leq \alpha < p)$$

We denote by $C_p(\alpha)$ the subclass of M_p consisting of functions which are meromorphically convex of order α .

In particular

$$S_1^*(\alpha) = S^*(\alpha), \quad C_p(\alpha) = C(\alpha)$$

$$S^*(0) = S^*, \quad C(0) = C$$

We note that $f(z) \in C_p(\alpha) \Leftrightarrow \frac{zf'(z)}{p} \in S_p^*(\alpha)$

$$S_p^*(\alpha) = CS_p^* \quad , \quad C_p(\alpha) \subset C_p \quad (0 \leq \alpha < p),$$

$$S^*(\alpha) \subset S^* \quad , \quad C(\alpha) \subset C \quad (0 \leq \alpha < p).$$

We get the details in Goodman [6], Duren [4], Srivastava and Owa [10].

Differentiating equation (1.1) q times we get

$$(1.4) \quad D^q f(z) = \frac{(-1)^p (p+q-1)!}{(p-1)! Z^{p+q}} + \sum_{k=p}^{\infty} \frac{k!}{(k-q)!} a_k z^{k-q} \quad (q \leq p)$$

or

$$\frac{(p-1)! (-1)^q z^q D^q f(z)}{(p+q-1)!} = \frac{1}{Z^p} + \sum_{k=p}^{\infty} \frac{(-1)^q (p-1)! k!}{(k-q)! (p+q-1)!} a_k z^k$$

Let

$$F_q(z) = \frac{(-1)^q (p-1)! z^q D^q f(z)}{(p+q-1)!}$$

Then $F_q(z) = \frac{1}{Z^p} + \sum_{k=p}^{\infty} b_k z^k \in M_p$

where $b_k = \frac{(-1)^q (p-1)! k!}{(k-q)! (p+q-1)!} a_k$

Clearly $F_0(z) = f(z)$ and $F_1(z) = \frac{-zf'(z)}{p}$

Now, we consider

$$F(z) = Z^{2p} F_q(z)$$

$$(1.5) \quad \therefore F(z) = z^p + \sum_{k=p}^{\infty} b_k z^{k+2p}$$

Differentiating equation (1.5) m times, we get

$$(1.6) \quad \begin{aligned} F^m(z) &= p(p-1) \dots (p-m+1) z^{p-m} + \sum_{k=p}^{\infty} (k+2p)(k+2p-1) \dots \\ &\quad (k+2p-m+1) b_k z^{k+2p-m} \\ F^m(z) &= \frac{p!}{(p-m)!} z^{p-m} + \sum_{k=p}^{\infty} \frac{(k+2p)!}{(k+2p-m)!} b_k z^{k+2p-m}, \quad (m < p), \end{aligned}$$

Several classes of meromorphically multivalent functions were studied by Srivastava, et.al. ([1], [2], [3], [7] and [9]) and by Dziok and Irmak ([5], [11]).

In this paper we obtain some sufficient conditions for functions to be in the class $M_p(m, q; \alpha)$ and $N_p(m, q; \alpha)$. Our results are given in the form of theorems.

In proving our results, we need the following Jack's lemma [8].

Lemma 1.1 : Let $w(z)$ be analytic in the unit disk U . $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point z_0 then $z_0 w'(z_0) = cw(z_0)$, where c is real and $c \geq 1$.

2. THE CLASS $M_p(m, q; \alpha)$:

Definition 2.1 : A function $f(z) \in M_p$ is said to be in class $M_p(m, q; \alpha)$ if it satisfies the inequality

$$(2.1) \quad \left| \frac{ZF^{m+1}(z)}{F^m(z)} - (p-m) \right| < p-m-\alpha \quad .$$

for $p \in \mathbb{N}$, $0 \leq \alpha < p-m$, $m < p$.

It is clear that

$$M_p(0, 0; \alpha) \equiv S_p^*(\alpha) , \quad (0 \leq \alpha < p)$$

$$M_p(0, 1; \alpha) \equiv C_p(\alpha) , \quad (0 \leq \alpha < p)$$

Theorem 2.1 : For $p > 1$, $M_p(m+1, q; \beta) \subset M_p(m, q; \alpha)$, where

$$(2.2) \quad 0 \leq \alpha \leq \frac{2p-2m+2+\beta-\sqrt{(\beta+2)^2+4(p-m)(p-m-1-\beta)}}{2}$$

Proof : Let $f(z) \in M_p(m+1, q; \beta)$. Then we have

$$(2.3) \quad \left| \left(\frac{ZF^{m+2}(z)}{F^{m+1}(z)} \right) - (p-m-1) \right| < p-m-1-\beta$$

Now, we define $w(z)$ by

$$(2.4) \quad \frac{ZF^{m+1}(z)}{F^m(z)} - (p-m) = (p-m-\alpha)w(z)$$

Clearly $w(z)$ is analytic in U and $w(0) = 0$. Differentiating equation (2.4) logarithmically, we get

$$(2.5) \quad \frac{ZF^{m+2}(z)}{F^{m+1}(z)} - (p-m-1) = (p-m-\alpha)w(z) \left[1 + \frac{zw'(z)}{w(z)} \cdot \frac{1}{[(p-m)+(p+m-\alpha)w(z)]} \right]$$

Now, suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1$$

By applying Jack's lemma, we have

$$(2.6) \quad z_0 w'(z) = c w(z_0), \quad w(z_0) = e^{i\theta} \quad (c \geq 1)$$

Then making use of (2.6) in (2.5), we obtain

$$\begin{aligned} \left| \frac{z_0 F^{m+2}(z_0)}{F^{m+1}(z_0)} - (p-m-1) \right| &= (p-m-\alpha) w(z_0) \left[1 + \frac{z_0 w'(z_0)}{w(z_0)} \cdot \frac{1}{[(p-m)+(p+m-\alpha)w(z_0)]} \right] \\ &= (p-m-\alpha) \left| 1 + \frac{c}{(p-m)+(p-m-\alpha)e^{i\theta}} \right| \\ &= (p-m-\alpha) \left\{ 1 + \frac{c}{p-m} \operatorname{Re} \left\{ \frac{1}{1+ae^{i\theta}} \right\} \right\}, \text{ where } a = \frac{p-m-\alpha}{p-m} \\ &= (p-m-\alpha) \left\{ 1 + \frac{c}{p-m} \left\{ \frac{1}{2 + \frac{a^2-1}{1+a\cos\theta}} \right\} \right\} \\ &\geq (p-m-\alpha) \left[1 + \frac{1}{(p-m)} \left\{ \frac{1}{2 + \frac{a^2-1}{1+a}} \right\} \right] \\ &= \frac{(p-m-\alpha)(2p-2m-\alpha+1)}{(2p-2m-\alpha)} \end{aligned}$$

By (2.2), we have

$$\left| \left(\frac{z_0 F^{m+2}(z_0)}{F^{m+1}(z_0)} \right) - (p-m-1) \right| \geq p-m-1-\beta$$

which contradicts (2.3). Hence $|w(z)| < 1$ and from (2.4), it follows that $f(z) \in M_p(m, q; \alpha)$.

Corollary 2.2 : For $p > 1$, $M_p(m+1, q; \alpha) \subset M_p(m, q; \alpha)$

$$0 \leq \alpha < p-m, \quad m < p.$$

Theorem 2.3 : If a function $f(z) \in M_p$ satisfies the inequality

$$(2.7) \quad \left| \frac{zF^{m+1}(z)}{F^m(z)} - (p-m) \right|^{\lambda} \left| \frac{zF^{m+2}(z)}{F^{m+1}(z)} - (p-m-1) \right|^{\beta} < (p-m-\alpha)^{\lambda+\beta} \left(\frac{2p-2m-\alpha+1}{2p-2m-\alpha} \right)^{\beta}$$

for $\beta \geq 0, \lambda \in \mathbb{R}$, then $f(z) \in M_p(m, q; \alpha)$.

Proof : Define $w(z)$ as we define in (2.4). On using (2.4) and (2.5), we consider

$$\begin{aligned} |H(z)| &= \left| \frac{zF^{m+1}(z)}{F^m(z)} - (p-m) \right|^{\lambda} \left| \frac{zF^{m+2}(z)}{F^{m+1}(z)} - (p-m-1) \right|^{\beta} \\ &= (p-m-\alpha)^{\lambda+\beta} |w(z)|^{\lambda+\beta} \left| 1 + \frac{zw'(z)}{w(z)} \cdot \frac{1}{\{(p+m)+(p-m-\alpha)w(z)\}} \right|^{\beta} \end{aligned}$$

Using Jack's lemma and (2.5) we get

$$\begin{aligned} |H(z_0)| &= (p-m-\alpha)^{\lambda+\beta} \left| 1 + \frac{c}{(p-m)(p-m-\alpha)e^{i\theta}} \right|^{\beta} \\ &\geq (p-m-\alpha)^{\lambda+\beta} \left(\frac{2p-2m-\alpha+1}{2p-2m-\alpha} \right)^{\beta} \end{aligned}$$

which contradicts (2.7), so we have $|w(z)| < 1$. Then from (2.4), we get the inequality (1.6), i.e. $f(z) \in M_p(m, q; \alpha)$.

If $\lambda = 0, \beta = 1$, we get

Corollary 2.4 : If a function $f(z) \in M_p$ satisfies the inequality

$$\left| \frac{zF^{m+2}(z)}{F^{m+1}(z)} - (p-m-1) \right| < (p-m-\alpha) \left(\frac{2p-2m-\alpha+1}{2p-2m-\alpha} \right)$$

then $f(z) \in M_p(m, q; \alpha)$.

If $m = 0$ and $q = 0, 1$ respectively, we get the following results.

Result 1 : A function $f(z) \in M_p$ satisfies

$$\left| \frac{z[z^{2p}f(z)]''}{[z^{2p}f(z)]'} + 1 - p \right| < (p-\alpha) \left(\frac{2p-\alpha+1}{2p-\alpha} \right)$$

then $f(z) \in S^*_p(\alpha)$.

Result 2 : A function $f(z) \in M_p$ satisfies

$$\left| \frac{z(z^{2p+1}f(z))''}{(z^{2p+1}f(z))'} + 1 - p \right| < (p - \alpha) \left(\frac{2p - \alpha + 1}{2p - \alpha} \right)$$

then $f(z) \in C_p(\alpha)$.

Theorem 3.1 : If a function $f(z) \in M_p$ satisfies the inequality

$$(3.1) \quad \operatorname{Re} \left[\frac{2zF^{m+1}(z)}{F^m(z)} - \frac{zF^{m+2}(z)}{F^{m+1}(z)} - 1 \right] > \frac{\alpha(2p - 2m - 2\alpha + 1)}{2(p - m - \alpha)}.$$

for $0 \leq \alpha < p - m$, then

$$\operatorname{Re} \left(\frac{zF^{m+1}(z)}{F^m(z)} \right) > \alpha.$$

Proof : Let

$$(3.2) \quad \frac{zF^{m+1}(z)}{F^m(z)} = (p - m - \alpha) \left(\frac{1 + w(z)}{1 - w(z)} \right) + \alpha$$

where $w(z)$ is analytic in U and $w(0) = 0$. It suffices to show that $|w(z)| < 1$ for all $z \in U$.

If there exists a point $z_0 \in U$ such that $|w(z)| < 1$ for $|z| < |z_0|$ and $|w(z_0)| = 1$, then from Jack's lemma [8],

we have $z_0 w'(z_0) = k w(z_0)$, $k \geq 1$ setting $w(z_0) = e^{i\theta}$, it follows from (3.2) that

$$\begin{aligned} 1 + \frac{z_0 F^{m+2}(z_0)}{F^{m+1}(z_0)} - \frac{z_0 F^{m+1}(z_0)}{F^m(z_0)} &= \frac{\frac{2(p - m - \alpha)z_0 w'(z_0)}{(1 - w(z_0))^2}}{(p - m - \alpha) \left[\frac{1 + w(z_0)}{1 - w(z_0)} \right] + \alpha} \\ &= -\frac{\frac{2(p - m - \alpha)k}{2(1 - \cos \theta)}}{(p - m - \alpha) \frac{2i \sin \theta}{2(1 - \cos \theta)} + \alpha} \\ &= \frac{-(p - m - \alpha)k}{\alpha(1 - \cos \theta) + i(p - m - \alpha)\sin \theta} \\ &= \frac{-(p - m - \alpha)k[(1 - \cos \theta)\alpha - i \sin \theta(p - m - \alpha)]}{\alpha^2(1 - \cos \theta)^2 + (p - m - \alpha)^2 \sin^2 \theta} \end{aligned}$$

Thus, we obtain

$$(3.3) \quad \operatorname{Re} \left\{ \frac{z_0 F^{m+2}(z_0)}{F^m(z_0)} - \frac{z_0 F^{m+1}(z_0)}{F^m(z_0)} + 1 \right\} = \frac{-\alpha(p-m-\alpha)(1-\cos\theta)k}{\alpha^2(1-\cos\theta)^2 + (p-m-\alpha)^2 \sin^2 \theta}$$

Letting $1 - \cos \theta = t$, $0 \leq t \leq 2$ and writing

$$g(t) = \frac{t}{\alpha^2 t^2 + (p-m-\alpha)^2 (2t-t^2)}$$

(3.3) may be written as

$$\operatorname{Re} \left\{ \frac{z_0 F^{m+2}(z_0)}{F^m(z_0)} - \frac{z_0 F^{m+1}(z_0)}{F^m(z_0)} + 1 \right\} = -\alpha(p-m-\alpha)kg(t)$$

But, a simple calculation shows that $g(t)$ takes its minimum value at $t = 0$, and

$$\lim_{t \rightarrow 0} g(t) = \frac{1}{2(p-m-\alpha)^2}$$

Therefore, we have

$$\operatorname{Re} \left\{ \frac{z_0 F^{m+2}(z_0)}{F^{m+1}(z_0)} - \frac{z_0 F^{m+1}(z_0)}{F^m(z_0)} + 1 \right\} \geq \frac{-\alpha}{2(p-m-\alpha)}$$

hence

$$\operatorname{Re} \left[\frac{z_0 F^{m+1}(z_0)}{F^m(z_0)} - \left\{ \frac{z_0 F^{m+2}(z_0)}{F^{m+1}(z_0)} - \frac{z_0 F^{m+1}(z_0)}{F^m(z_0)} + 1 \right\} \right] \leq \alpha + \frac{\alpha}{2(p-m-\alpha)}$$

or

$$\operatorname{Re} \left\{ \frac{2z_0 F^{m+1}(z_0)}{F^m(z_0)} - \frac{z_0 F^{m+2}(z_0)}{F^{m+1}(z_0)} - 1 \right\} \leq \frac{\alpha(2p-2m-2\alpha+1)}{2(p-m-\alpha)}$$

This contradicts (3.1) and therefore, we have $|w(z)| < 1$ in U . This shows that

$$\operatorname{Re} \left\{ \frac{z F^{m+1}(z)}{F^m(z)} \right\} > \alpha \quad \text{in } U.$$

This completes the proof of the theorem.

4. THE CLASS $N_p(m, q; \alpha)$:

Definition 4.1 : A function $f(z) \in M_p$ is said to be in the class $N_p(m, q; \alpha)$ if it satisfies the inequality

$$(1.8) \quad \left| \left(\frac{zF^{m+1}(z)}{F^m(z)} \right)^{-1} - (p-m)^{-1} \right| < (p-m-\alpha)^{-1}$$

for $p \in \mathbb{N}$, $0 \leq \alpha < p-m$, $m < p$.

Theorem 4.1 : If a function $f(z) \in M_p$ satisfies the inequality

$$(4.1) \quad \operatorname{Re} \left[1 + \frac{zF^{m+2}(z)}{F^{m+1}(z)} - \frac{zF^{m+1}(z)}{F^m(z)} \right] > -\frac{(p-m)}{(2p-2m-\alpha)}$$

then $f(z) \in N_p(m, q; \alpha)$.

Proof : Let

$$(4.2) \quad \left(\frac{zF^{m+1}(z)}{F^m(z)} \right)^{-1} - (p-m)^{-1} = (p-m-\alpha)^{-1} w(z)$$

then $w(z)$ is analytic in U with $w(0) = 0$. Differentiating logarithmically, we have

$$\begin{aligned} - \left[\frac{1}{z} + \frac{F^{m+2}(z)}{F^{m+1}(z)} - \frac{F^{m+1}(z)}{F^m(z)} \right] &= \frac{(p-m-\alpha)^{-1} w'(z)}{(p-m-\alpha)^{-1} w(z) + (p-m)^{-1}} \\ - \left[\frac{1}{z} + \frac{F^{m+2}(z)}{F^{m+1}(z)} - \frac{F^{m+1}(z)}{F^m(z)} \right] &= \frac{(p-m-\alpha)^{-1} w'(z)}{(p-m)^{-1} \left[1 + \left(\frac{p-m-\alpha}{p-m} \right)^{-1} w(z) \right]} \\ &= \frac{bw'(z)}{1 + bw(z)} \end{aligned}$$

$$\text{where } b = \left(\frac{p-m}{p-m-\alpha} \right) \geq 1.$$

Thus,

$$-\operatorname{Re} \left[1 + \frac{zF^{m+1}(z)}{F^m(z)} - \frac{zF^{m+1}(z)}{F^m(z)} \right] = \operatorname{Re} \left(\frac{zbw'(z)}{1 + bw(z)} \right)$$

If we suppose that there exists a point $z_0 \in U$ such that $\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1$, then from Jack's lemma, we have

$$z_0 w'(z_0) = c w(z_0), \quad w(z_0) = e^{i\theta}, \quad (c \geq 1)$$

$$\begin{aligned} \therefore -\operatorname{Re} \left[1 + \frac{z_0 F^{m+2}(z_0)}{F^{m+1}(z_0)} - \frac{z_0 F^{m+1}(z_0)}{F^m(z_0)} \right] &= \operatorname{Re} \left(\frac{bce^{i\theta}}{1 + be^{i\theta}} \right) \\ &= c \operatorname{Re} \left(1 - \frac{1}{1 + be^{i\theta}} \right) \end{aligned}$$

$$= c \left[1 - \operatorname{Re} \left\{ \frac{1}{1 + be^{i\theta}} \right\} \right]$$

$$\text{As } \operatorname{Re} \left\{ \frac{1}{1 + be^{i\theta}} \right\} = \frac{1}{2 + \frac{b^2 - 1}{1 + b \cos \theta}}; (b \geq 1)$$

$$\leq \frac{1}{1 + b},$$

We have

$$-\operatorname{Re} \left[1 + \frac{z_0 F^{m+2}(z_0)}{F^{m+1}(z_0)} - \frac{z_0 F^{m+1}(z_0)}{F^m(z_0)} \right] \geq c \left\{ 1 - \frac{1}{b+1} \right\}$$

$$\geq \frac{p-m}{2p-2m-\alpha}$$

or

$$\operatorname{Re} \left\{ 1 + \frac{z_0 F^{m+2}(z_0)}{F^{m+1}(z_0)} - \frac{z_0 F^{m+1}(z_0)}{F^m(z_0)} \right\} \leq - \left\{ \frac{p-m}{2p-2m-\alpha} \right\}$$

which contradicts (4.1). Therefore, we have $|w(z)| < 1$. From (4.2), it follows that $f(z) \in N_p(m, q; \alpha)$.

If $m = 0$, $p = 1$ and $q = 0$, we have

Corollary 3.2 : If a function $f(z) \in M_p$ satisfies the inequality

$$\operatorname{Re} \left[1 + \frac{z(z^2 f(z))''}{(z^2 f(z))'} - \frac{z(z^2 f(z))'}{(z^2 f(z))} \right] > \frac{-1}{2-\alpha}$$

then $f(z) \in N(0, 0; \alpha)$.

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